The Electronic Journal of Mathematics and Technology, Volume 4, Number 3, ISSN 1933-2823

# Three Ladders-Walls Problems

*Wei-Chi YANG*
wyang@radford.edu
Department of Mathematics and Statistics
Radford University
Radford, VA 24142
USA

*Matthias KAWSKI*
kawski@asu.edu
School of Mathematical &
Statistical Sciences
Arizona State University
Tempe, Arizona 85287-1804
USA

**Abstract**

*In this paper, we solve three ladders-walls problems. The one ladder problem is a typical first year Calculus problem, see [3],. The extension to more ladders and walls using Lagrange Multipliers was discussed in [1], but the calculations were complex. A simplier solution to two ladders-walls problems were discussed in [2]. In this paper, we discuss the three ladders-walls problems and the extensions with the help of Dynamic Geometry and Computer Algebra System.*

## 1 Introduction

In practical terms, consider three parallel walls of with given constant heights $h_1, h_0, h_2 \geq 0$ standing on level ground given distances $d_1 \geq 0$ and $d_2 \geq 0$ apart. The task is to use three ladders (connected end to end) of combined minimum length to build a bridge over the three walls. Each ladder is assumed to have negligible thickness, and each is allowed to be vertical, in particular to be coincident with a wall. In mathematical terms, the problem is to find the shortest polygonal path in the upper half plane that consists of at most three line segments, which starts and ends on the rays $I_1 = (-\infty, -d_1) \times \{0\}$ and $I_2 = (d_2, \infty) \times \{0\}$, and which does not transversely intersect any of the line segments $\{-d_1\} \times [0, h_1)$, $\{0\} \times [0, h_0)$, and $\{d_2\} \times [0, h_2)$. Any polygonal path that satisfies these conditions will be called *admissible*, and the task is to find the admissible paths of minimum total length.

It is intuitively clear that for any given data it is always possible to bridge all three walls by using only two ladders, and that by using a third ladder of nonzero length one always can obtain a strictly smaller minimum for the combined length: One ladder must be non-vertical and have one end on one of the ground, (without loss of generality) at $(x_1, 0) \in I_1$ and the other end at $(\xi_1, \eta_1) \in \mathbb{R}^2$ with $\eta_1 > 0$ and $\xi_1 > x_1$.

It is immediately clear that for any $t \in (x_1, \min\{d_1, \xi_1\})$ one obtains a better (shorter) bridge by replacing the segment from $(x_1, 0)$ to $(t, \dfrac{t - x_1}{\xi_1 - x_1}\eta_1)$ by a vertical ladder from $(t, 0)$ to $(t, \dfrac{t - x_1}{\xi_1 - x_1}\eta_1)$. Thus one may expect that commonly the optimal three ladder solution might include a vertical ladder on one side.

This article investigates the dependence of the optimal solution on the given parameters, and, in particular, whether for every set of strictly positive parameters $d_i$ and $h_i$ the optimal solution always includes one vertical ladder. This article is motivated in part by familiar optimization problems in calculus texts that involve one ladder such as in [3], and problems were extended by the article using Lagrange Multipliers in [1]. The article [2] analyzes a two ladders and two walls using different approaches. This article is a generalization of [2], and it suggests one possible way to rephrase a generalized $n$-ladder-$n$-walls problems as a dynamic programming problem.

The first part of the article, in section 2, employs the Computer Algebra System MAPLE to analyze the problem, using a combination of symbolic and numeric techniques. The key innovation is to cast the multi-parameter optimization problem as nested one-dimensional optimization problems. The first of which is easily solved in closed form using computer algebra. Its solution is then passed to the next one-dimensional problem, which then is more naturally analyzed numerically for specific numeric data for the parameters $d_i$ and $h_i$. This use of computer algebra to reduce the complexity of the problem is critical for a successful analysis, and is a clear improvement over directly attacking typically a two dimensional optimization problem with five parameters (four after normalization).

The second part of the article, in section 3, is to deal with a special case when one outside ladder is vertical, the middle ladder is resting on the tip of the middle wall and the wall with the vertical ladder, we are asking where to place the last grounded ladder when we vary the height of the last wall and the distance between the last and the middle walls.

# 2 Analysis using computer algebra

## 2.1 Definitions and preliminary arguments

Consider the labeling of the following diagram with vertical *walls* standing at $A' = (-d_1, 0)$, $E' = (0, 0)$, and $D' = (d_2, 0)$ whose tops are located at $A = (-d_1, h_1)$, $E = (0, h_0)$, and $D = (d_2, h_2)$ respectively. We assume that all $d_i, h_i$ are strictly positive – the degenerate cases are elementary.

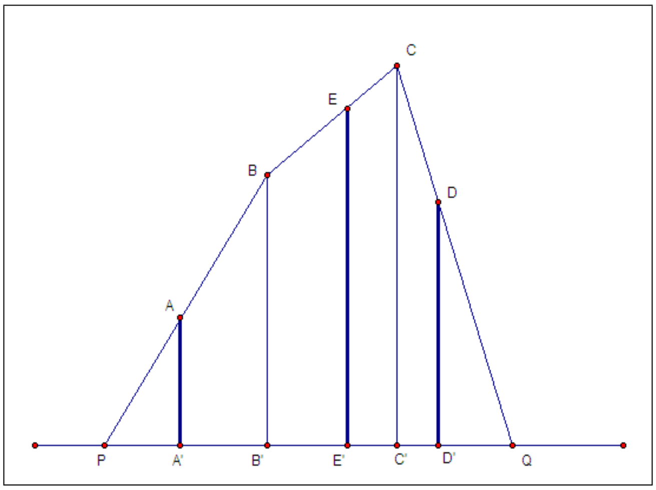



Figure 1

The three ladders are connected end to end and define the polygonal path connecting $P = (-x_1, 0)$, $B = (\xi_1, \eta_1)$, $C = (\xi_2, \eta_2)$, to $Q = (x_2, 0)$.

The problem requires that

$$x_1 \leq -d_1 \text{ and } x_2 \geq d_2. \tag{1}$$

Arguing similarly as in the introduction, it is clear that if $\xi_1 < -d_1$ or $\xi_2 > d_2$, then the corresponding path cannot be optimal. In order to be admissible the path must be such that each of the *outside ladders* either clears the outside wall or rests on top of it.

$$\frac{\eta_1}{\xi_1 - x_1} \leq \frac{\eta_1 - h_1}{\xi_1 + d_1} \text{ and } \frac{\eta_2}{x_2 - \xi_1} \leq \frac{\eta_2 - h_2}{d_2 - \xi_2} \tag{2}$$

In the case of strict inequality, it is clear that by holding the point $(\xi_i, \eta_i)$ fixed and moving the foot $(x_i, 0)$ closer to the wall one obtains another admissible path with shorter total length. Hence it is clear that for any optimal solution equality must hold in each equation, i.e.

$$\frac{\eta_1}{\xi_1 - x_1} = \frac{\eta_1 - h_1}{\xi_1 + d_1} \text{ and } \frac{\eta_2}{x_2 - \xi_1} = \frac{\eta_2 - h_2}{d_2 - \xi_2} \tag{3}$$

Regarding the middle wall and middle ladder there are numerous possible scenarios. In the case that the top $E = (0, h_0)$ of the middle wall does not lie above the line connecting the tops $A = (-d_1, h_1)$ and $D = (d_2, h_2)$ of the outside walls, it is immediate that the optimal solution consists of the polygonal path made up of two vertical ladders $A'A$ and $D'D$ connected by the middle ladder $AD$ resting on the tops of the outside walls. Thus henceforth we assume that the top $E$ of the middle wall lies strictly above the line segment $AD$ which is equivalent to the inequality

$$\frac{h_0 - h_2}{d_2} < \frac{h_0 - h_1}{d_1}. \tag{4}$$

We argue that (under this assumption) in any optimal solution the middle ladder must rest on the top of the middle wall. We first need to rule out the case that $\xi_1 \xi_2 > 0$. Thus suppose (without loss of generality) that $\xi_1 < \xi_2 < 0$. Keeping the left endpoint $B = (\xi_1, \eta_1)$ unchanged, it is clear that a shorter path is obtained by replacing the point $C = (\xi_2, \eta_2)$ where the middle and the third ladders connect by the new point $\tilde{C} = (0, \frac{x_2}{x_2 - \xi_2}\eta_2)$. Pictorially, the middle ladder is lengthened and rotated about its left end point, and the right ladder is only shortened. Hence that operation does not affect any of the other inequality constraints that the right ladder must satisfy.

Thus henceforth we assume that

$$-d_1 \leq \xi_1 \leq 0 \leq \xi_2 \leq d_2. \tag{5}$$

Finally we note that if the middle ladder lies strictly above the top of the middle wall, then a shorter overall solution can be obtained by using by lowering it and lengthening it: for example, as above rotate the middle ladder about its left endpoint while simultaneously lengthening it and shortening the right ladder, in other words, increasing $\xi_2$. Thus we only need to consider the case when the top of the middle wall $E = (0, h_2)$ lies on the line segment connecting $B = (\xi_1, \eta_1)$ and $C = (\xi_2, \eta_2)$. This is equivalent to the equality

$$\frac{h_0 - \eta_1}{-\xi_1} = \frac{h_0 - \eta_2}{\xi_2}. \tag{6}$$

## 2.2 The optimization problem and solution strategy

For any fixed set of strictly positive parameters $h_1, h_0, h_2, d_1, d_2 > 0$ that satisfy the concavity constraint (4) the objective is to minimize the length of the polygonal path $PBCQ$ subject to the above inequality constraints (1), (5), and equality constraints (3) and (6). Initially the path involves the six variables $x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2$. But the equality constraints (3) and (6). allow on to reduce the problem to three variables, and one has many choices.

Using the invariance of the problem under scaling, one may fix one of the parameters, e.g. the height of the middle wall $h_0 = 1$ and is left with the problem of minimizing a four-parameter family of scalar functions of three variables. Assuming that one knows that one of the outside ladders in an optimal solution must be vertical, one still has for every fixed choice of parameters $d_i$ and $h_i$ a two dimensional optimization problem that may be explored or solved numerically.

A much simplifying choice for an analytic exploration and solution using computer algebra is to focus on the slope of the middle ladder as the main variable of interest

$$\text{slope } m = \frac{\eta_2 - \eta_1}{\xi_2 - \xi_1} = \frac{\eta_1 - h_0}{\xi_1} = \frac{\eta_2 - h_0}{\xi_2} \tag{7}$$

Arguing analogously to the preceding section it is easy to see that this slope $m$ must, in any optimal solution, satisfy the inequalities

$$\frac{h_2 - h_0}{d_2} \leq m \leq \frac{h_0 - h_1}{-d_1}. \tag{8}$$

Else the polygonal path between the walls would have a piece that is not concave, and as before could be shortened. For simplicity, we denote

$$m_{\max} = \frac{h_0 - h_1}{-d_1}, \text{ and } m_{\min} = \frac{h_2 - h_0}{d_2}. \tag{9}$$

The key idea in our solution is to consider this slope $m$ an additional parameter, and consider two decoupled parameterized families of one-dimensional optimization problems: For parameters $d_i$ and $h_i$, and an additional parameter $m$ minimize the lengths of each of the paths $\pi_1 = PBE$ and $\pi_2 = EDQ$. That is for given $m$ satisfying (8) find an algebraic expression for the optimal choices of $x_1, \xi_1, \eta_1$ and $x_2, \xi_2, \eta_2$, respectively. Due to the equality constraints (3) and (6) each member of either triple determines the other two. The particular choice seems to make little difference – maybe choosing the slopes of the outside ladders would be a more consistent choice, but as the accompanying MAPLE worksheet demonstrates, the location $x_i$ of the bottom end of the outside ladders makes is both practical and convenient.

In summary, the strategy is to find the optimal choices $x_i^*(m)$ that for given slope $m$ minimize the lengths of the paths $\pi_i$. In the sequel minimize the sum of these two paths over the slope $m$.

## 2.3 Optimizing outside ladders for given middle slope

This section summarizes the main formula and results for the parametric one dimensional optimization problems for the outside ladders for any given slope of the middle ladder. Details of the calculations are available in the accompanying MAPLE worksheets, see [10,11]. The length of each half path $\pi_i$

$$s_i = \mathrm{Length}(\pi_i) = \sqrt{\eta_i^2 + (\xi_i - x_i)^2} + \sqrt{(h_0 - \eta_i)^2 + \xi_i^2} \tag{10}$$

can be expressed as a function of a single scalar variable $x_i$ using the equality constraints (3) and (6). In the sequel we only list the results for the left half $i = 1$, the results for the right side are analogous except for a minor change in notation due to our choice of naming the $x$-coordinates of the bottoms of the outside walls $-d_1$ and $+d_2$.

$$s_1 = \sqrt{(-h_0 x_1 + h_0 d_1 + h_1 x_1)^2 (m^2 + 1)/(m x_1 - m d_1 + h_1)^2} \tag{11}$$
$$+ \sqrt{((x_1^2 - 2x_1 d_1 + d_1^2 + h_1^2)(m x_1 + h_0)^2/(m x_1 - m d_1 + h_1)^2} \tag{12}$$

Using elementary calculus, it is straightforward, but forbiddingly tedious by hand, to calculate the derivative of $s_1$ as a function of $x_1$ and solve for its roots. Using computer algebra, these are returned as the roots of a quartic polynomial whose coefficients are quadratic polynomials in $m$.

$$\begin{aligned} x_1^*(m) = \mathrm{RootOf}(&m^2 Z^4 + (-4d_1 m + 2h_1) m Z^3 \\ &+ (6d_1^2 m^2 - 4d_1 h_1 m + (2h_0 h_1 - h_1^2)) Z^2 \\ &+ (-(4d_1^3 + 2d_1 h_1^2) m^2 + (2d_1^2 h_1 - 2h_1^2 h_0 + 2h_1^3) m - 4h_0 d_1 h_1) Z \\ &+ (h_1^2 d_1^2 + d_1^4) m^2 + (2h_0 h_1 d_1^2 - h_0^2 h_1^2 - d_1^2 h_1^2 - h_1^4 + 2h_0 h_1^3))). \end{aligned} \tag{13}$$

For any given numerical values for admissible parameters $d_i, h_i$, the computer algebra system easily evaluates the quartic polynomial and its the roots as functions of the slope $m$. It takes only little manual intervention to select the real roots, and among these, the ones which satisfy the constraints. Of particular interest is the question: when is the function $s_1$ monotone on the admissible interval. Recall the concavity requirement that the left ladder have a slope that is not smaller than the slope $m$ of the middle ladder, i.e. $\frac{h_1}{-d_1 - x_1} \geq m$ together with the basic hypothesis $x_1 < -d_1$ restricts the domain of interest of $s_1$ to the interval

$$d_1 - \frac{h_1}{m} \leq x_1 \leq -d_1. \tag{14}$$

We recall from (9) that the largest slope for the middle ladder is denoted by $m_{\max}$, which occurs when the middle ladder connects the tops of the left and middle walls; in such case, we have left vertical ladder. Similarly, we denote $m_{\min}$ by the smallest slope for the middle ladder when the middle ladder connects the tops of right ladder and middle ladder or we have right vertical ladder. We observe

$$x_{1\min} = d_1 - \frac{h_1}{m_{\max}} = \frac{h_0 d_1}{h_0 - h_1}, \text{ and } x_{1\max} = -d_1. \tag{15}$$

The optimal solution is a critical point of $s_1$ inside the interval $(x_{1\min}, x_{1\max})$, or is attained at one of the endpoints if $s_1$ is monotone on this interval. The endpoint minimum $x_1^* = x_{1\min}$ corresponds to the left ladder resting on top of the left and middle walls, $(\xi_1, \eta_1) = (0, h_0)$ or $B = E$. The endpoint minimum $x_1^* = x_{1\max} = -d_1$ corresponds to the middle ladder resting on top of the left and middle walls, $(\xi_1, \eta_1) = (d_1, h_1)$ or $A = B$, and the left ladder vertical. Analogous to (15), we have

$$x_{2\min} = d_2, \text{ and } x_{2\max} = \frac{h_0 d_2}{h_0 - h_2}. \tag{16}$$

The following identities are helpful in Examples 1 and 2. For $B = (\xi_1 \eta_1)$, and $C = (\xi_2 \eta_2)$, we have

$$\xi_1 = \frac{-h_0 x_1 + h_0 d_1 + h_1 x_1}{m x_1 - m d_1 + h_1}, \tag{17}$$

$$\xi_2 = \frac{-h_0 x_2 + h_0 d_2 + h_1 x_2}{m x_2 - m d_2 + h_2}, \tag{18}$$

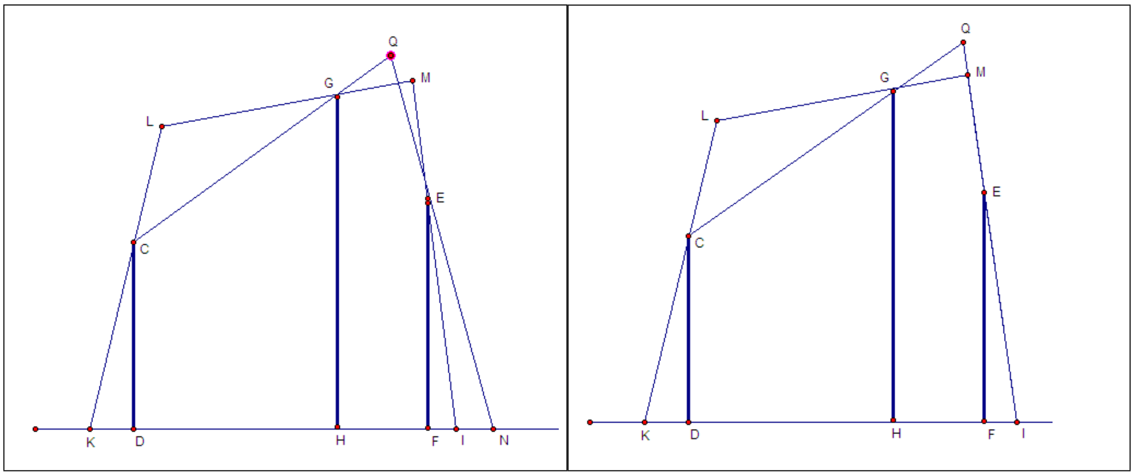
$$\eta_1 = \frac{h_1 (x_1 - \xi_1)}{x_1 - d_1}, \text{ and} \tag{19}$$

$$\eta_2 = \frac{h_2 (x_2 - \xi_2)}{x_2 - d_2}. \tag{20}$$

A particularly nice animation shows for each admissible value of the slope $m$ of the middle ladder, the suboptimal constellation of three ladders that optimizes the lengths of both the half-polygons $\pi_1$ and $\pi_2$. See the accompanying worksheet [10] for details.

## 2.4 One outside ladder must be vertical

Consider the three ladders-walls diagram of Figure 2(a) :




Figures 2(a) and 2(b). One outside ladder should be vertical.

Without loss of generality, we assume the heights of two outside walls are not equal ($h_1 \neq h_2$). Furthermore, we assume $DH > HF$ or $|d_1| > d_2$. We consider two cases, first we have a left vertical ladder on $CD$ and consider the total length of $DCQN$. In this case, we denote the sum of three ladders by $s_1^1 + s_2^1$. Next we consider the non vertical ladder case $KLMI$, and denote the sum of the three ladders by $s_1^2 + s_2^2$. We want to show that if we are given $h_1 = CD, h_0 = GH$ and $EF = h_2$, for any (fixed) non-vertical scenario $KLMI$, we can always find a vertical scenario $DCQN$ such that $s_1^1 + s_2^1 < s_1^2 + s_2^2$ or equivalently,

$$s_1^2 - s_1^1 > s_2^1 - s_2^2. \tag{21}$$

We note that the $x$ coordinate of $M$ should be in the range of $[0, d_2]$. By using a dynamic geometry software, we make the following observations:

- The inequality (21) holds obviously if $Q = M = G$, see Figure 3 below:

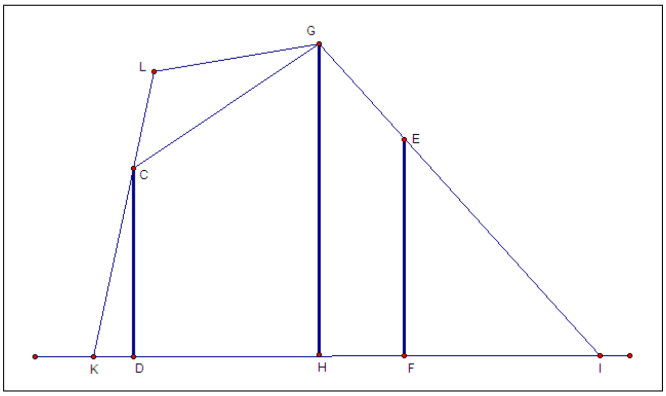



Figure 3. A special case.

- Consider Figure 2(a), given any fixed non-vertical scenario $KLMI$,we move the point $Q$ so that $Q, M$ and $E$ are colinear, then $N = I$ and we have the scenario of Figure 2(b). In other words, we have
$$s_2^1 - s_2^2 = (GQ + QM) - GM \tag{22}$$
when considering the triangle $GQM$.

- We note
$$s_1^2 = KL + LG > DC + CG = s_1^1, \tag{23}$$
by applying the triangular inequalities on the triangles $KCD$ and $CLG$ respectively. We claim that the difference, $s_1^2 - s_1^1$, is always larger than the difference, $s_2^1 - s_2^2$ since $DH < HF$ and $CD < EF$. Other cases can be proved analogously. In other words, we have $s_1^2 - s_1^1 > s_2^1 - s_2^2$ or $s_1^2 + s_2^2 > s_1^1 + s_2^1$.

## 2.5 Optimizing the slope of the middle ladder

It is straightforward, when using a computer algebra system, to evaluate the lengths $s_i$ at the optimal placements $x_i^*(m)$ of the outside ladders. While the resulting expression for

$$(\text{total length})(m) = (s_1 \circ x_1^*)(m) + (s_2 \circ x_2^*)(m) \tag{24}$$

is no longer amenable to symbolic analysis without any further innovations, it is, for fixed numerical parameter values $d_i, h_i$ a simple scalar function of a single real variable $m$, and it is straightforward to minimize it numerically. It is instructive to inspect animations of any of the above that vary any one of the parameters above. We summarize the steps of how we use MAPLE for computation as follows:

1. Solve $\dfrac{\partial s_i}{\partial x_i} = 0$ for $x_i$:
   (a) $x_i$ is the root of a polynomial of $m$ of degree 4.
   (b) $x_1$ is a function of $m, d_1, h_1$ and $h_0$.

(c) $x_2$ is a function of $m, d_2, h_2$ and $h_0$.

2. Substitute all data $(h_i, d_i)$ into $x_i(m)$ to get $x_i^*(m)$ : For any given slope of the middle slope, we get the critical points for $x_i^*(m)$.

   (a) Substitute all numerical data and $x_i^*(m)$ into $s_i$ and plot $s_i(x_i^*(m))$.

   (b) We consider the graph of $s_1 + s_2$; note the graph of $s_1 + s_2$ gives us a conjecture where we should use left or right vertical ladder, see MAPLE worksheets in [10] and [11] respectively.

3. Steps of finding the optimal solution:

   (a) Determine if we should use left vertical or right vertical.

   (b) If it is left vertical, the slope of middle ladder $m$ is equal to its maximum, $m_{\max}$.

   (c) If it is right vertical, the slope of the middle ladder $m$ is equal to its minimum, $m_{\min}$.

4. For the left vertical case (see Example 1), since $m = m_{\max}$, we know $x_1 = d_1$.To figure out $x_2$,we substitute all data and $m = m_{\max}$ into $s_2$, and plot the function $s_2$. If $s_2$ is decreasing with respect to $x_2$, then the minimum happens at $x_2 = x_{2\max}$. Otherwise, we find the critical point $x_2$ by using $\dfrac{\partial s_2}{\partial x_2} = 0$.

5. For the right vertical case (see Example 2), since $m = m_{\min}$, we know $x_2 = d_2$.To figure out $x_1$, we substitute all data and $m = m_{\min}$ into $s_1$, and plot the function $s_1$. If $s_1$ is increasing with respect to $x_1$, then the minimum happens at $x_1 = x_{1\min}$. Otherwise, we find the critical point $x_1$ by using $\dfrac{\partial s_1}{\partial x_1} = 0$.

**Example 1** *(We refer to Figure 1.) We are given a set of numerical data for the heights of three respective ladders and the widths between two ladders. In this case, we let* $h_1 = \dfrac{165}{100}, h_2 = \dfrac{225}{100}, h_0 = \dfrac{295}{100}, d_1 = -\dfrac{205}{100}$ *and* $d_2 = \dfrac{70}{100}$*. We shall minimize the total length of* $PBCQ$*.*

The detailed computations can be found in the MAPLE worksheet [10]. We first note the function $s_1$ and $s_2$ as follows:

1. When we substitute the given numeric values for $h_1, h_2, h_0, d_1$ and $d_2$ into $s_1$, and $s_2$ respec-

tively. With the help of MAPLE, we have $s_1$ and $s_2$ as functions of two variables:

$$s_1 = \sqrt{\frac{(1+m^2)\left(-\frac{13x_1}{10}-\frac{2419}{400}\right)^2}{\left(mx_1-\frac{41}{20}m+\frac{33}{20}\right)^2}} + \sqrt{\frac{(x_1^2+\frac{41}{10}x_1+\frac{277}{40})\left(mx_1+\frac{59}{20}\right)^2}{\left(mx_1-\frac{41}{20}m+\frac{33}{20}\right)^2}}, \text{ and} \tag{25}$$

$$s_2 = \sqrt{\frac{(1+m^2)\left(-\frac{7x_2}{10}-\frac{413}{200}\right)^2}{\left(mx_2-\frac{7}{10}m+\frac{9}{4}\right)^2}} + \sqrt{\frac{(x_1^2+\frac{7}{5}x_2+\frac{2221}{400})\left(mx_2+\frac{59}{20}\right)^2}{\left(mx_1-\frac{7}{10}m+\frac{9}{4}\right)^2}} \tag{26}$$

2. We follow the steps layed out in Section 2.5. We set $\frac{\partial s_1}{\partial x_1}=0$, $\frac{\partial s_2}{\partial x_2}=0$, and properly select the respective critical points $x_1^*(m)$ and $x_2^*(m)$. We next sketch the graphs (numerically) of $s_1(x_1^*(m))$, $s_2(x_2^*(m))$ and $s_1(x_1^*(m))+s_2(x_2^*(m))$ in blue, red and black respectively below:

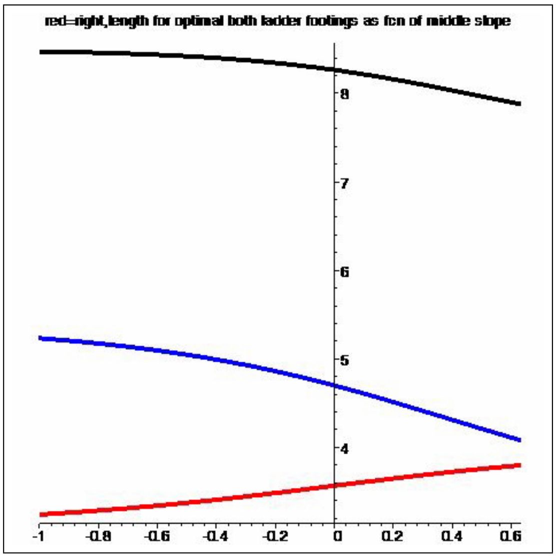



Figure 4. Example 1.

Our analysis confirms our finding from the picture above that the minimum of $s_1(x_1^*(m))+s_2(x_2^*(m))$ occurs when $m$ is at its maximum. Or equivalently, when the left ladder is vertical. In this case, we have $B=E$, and we can reduce $s_2$ to be a function of $x_2$. Applying $\frac{ds_2}{dx_2}=0$, we find the minimum for $s_2$ is at $x_2=1.721233899$ and we find $C=(\xi_2\eta_2)=(0.296843175, 3.138242015)$. The the shortest total length for $PBCQ$ is about $7.875313261$.The diagram can be seen below:

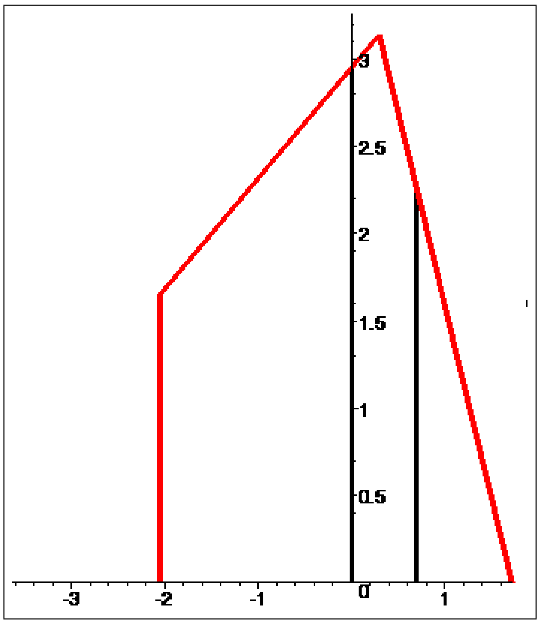


Figure 5. Optimal solution for Example 1

**Example 2** *Consider the following three ladders-walls problem, where the heights of three walls are given as follows:* $h_1 = \frac{75}{100}, h_0 = \frac{37}{20}$, *and* $h_2 = \frac{53}{40}$ *respectively, the width between* $h_1$ *and* $h_0$ *(which we call it* $d_1$*) is* $\frac{6}{5}$, *and the width between* $h_0$ *and* $h_2$ *(which we call it* $d_2$*) is* $\frac{151}{200}$.

The detailed computations can be found in the MAPLE worksheet [11]. We follow the strategy described in Section 2.5, and plot the $s_1(x_1^*(m)), s_2(x_2^*(m))$ and $s_1(x_1^*(m)) + s_2(x_2^*(m))$ in blue, red and black respectively in Figure 6 below:

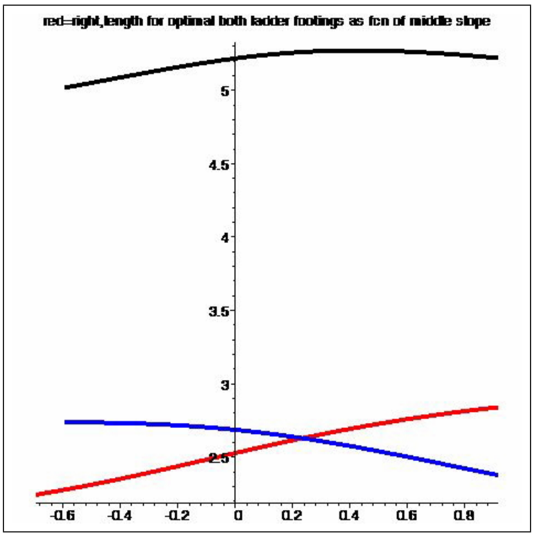



Figure 6. Example 2

We follow the steps described in Section 2.5. The optimal solution for the shortest total length for $s_1 + s_2$ in this case is at $m = m_{\min}$ or we have right vertical ladder. We also see that MAPLE worskheet [11] that the function $s_1(x_1(m_{\min})$ is an increasing function of $x_1$ in $[x_{1\,\min}, d_1]$, thus the optimal solution for the shortest total length for $s_1 + s_2$ is at $x_1 = x_{1\,\min}$ and $x_2 = d_2$ respectively, and the optimal solution is $4.982393946$. We note that this is a special case where middle ladder $DE$ touches the left outside ladder $PE$ at the tip of the middle wall $E$, see Figure 7 below.

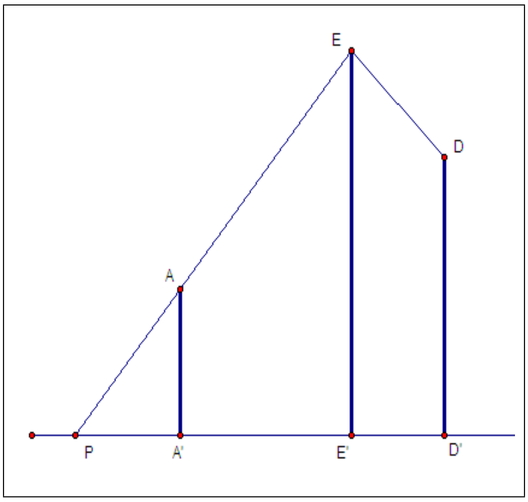



Figure 7. Optimal solution for Example 2

## 2.6 When will the second ladder touch the top of the middle wall?

In the two ladders problem discussed in [2], author describes a condition where one ladder has to be vertical, whose length is exactly the height of the taller wall; this leads to the second ladder touching the tips of both walls. For example, let us consider the Figure 8 below by considering two walls $AB$ and $EL$. We denote the heights of $AB$ and $EL$ as $h_1$ and $h_0$ respectively. We assume $h_1 < h_0$, and denote the width $BL$ as $d$. Then if

$$d < (h_0 - h_1)\sqrt{\left(\frac{h_0}{h_1}\right)^2 - 1}, \tag{27}$$

as described in [2], then we need to place one vertical ladder on $EL$ to achieve the optimal solution. In other words, we have two ladders $EL$ and $EF$ which touch the tips of two walls, $EL$ and $AB$, respectively. Consider the Figure 8 below again for three ladders-walls problem; the three walls are $AB, EL$ and $HI$ respectively. We consider two cases here, one is $CEHI$, where three ladders $CE, EH$ and $HI$ are connected at the tips of three walls, $AB, EL$ and $HI$, respectively. The second case is $FGHI$, where the second ladder $GH$ is not connected at the tip of $EL$. We demonstrate that the condition (27) cannot be used on both widths $BL$ and $LI$ in three ladders-walls problem, it is sufficient if we can demonstrate one example such that $FG + GE < CE$.

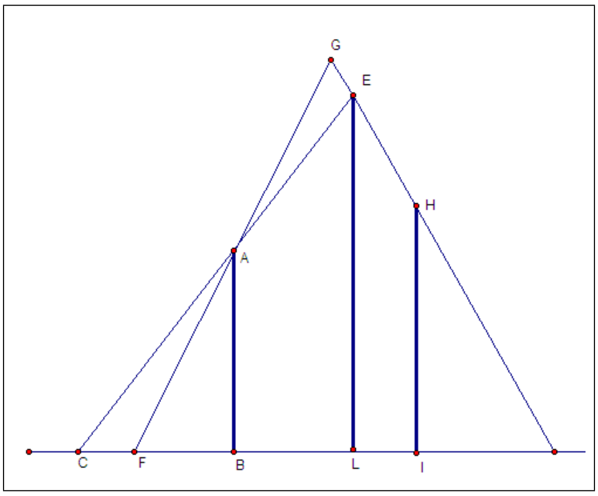



Figure 8

We fix the heights of three walls and note the followings:

1. By moving the point $C$ to the right, we reduce the width $BL$. The length $CE$ get closer to the total length of $FG + GE$.

2. By moving the point $C$ to the left, we enlarge the wide $BL$. For example, we let $h_0 = 1.358811, h_1 = 0.83$ and $h_2 = 0.95$. Note that

$$(h_0 - h_1)\sqrt{\left(\frac{h_0}{h_1}\right)^2 - 1} = 0.685451545, \tag{28}$$

$$(h_0 - h_2)\sqrt{\left(\frac{h_0}{h_2}\right)^2 - 1} = 0.4180752379 \tag{29}$$

we find $F = (-1.122829, 0)$ such that $FG + GE = 1.835268 < CE = 1.8432622$.

**Discussions**: We now describe a numerical condition where the second ladder touches the top of the middle and one of the outside walls below. Due to the number of variables involved, we do not expect to find a solution satisfying a symbolic inequality; instead, we describe a numerical solution below.

1. The three ladders-walls is a **right vertical** case for which the second ladder connects the tops of the middle and right walls if the following conditions are met simultaneously.

   (a) $s_1(x_1(m_{\min})) + s_2(x_2(m_{\min})) < s_1(x_1(m_{\max})) + s_2(x_2(m_{\max}))$ and

   (b) $\dfrac{d}{dx_1}(s_1(x_1(m_{\min})) > 0$ in $[x_{1\,\min}, d_1]$. In other words, $s_1(x_1(m_{\min})$ is an increasing function of $x_1$ in $[x_{1\,\min}, d_1]$.

2. The three ladders-walls is a **left vertical** case for which the second ladder connects the tops of the middle and left walls if the following conditions are met simultaneously:

   (a) $s_1(x_1(m_{\max})) + s_2(x_2(m_{\max})) < s_1(x_1(m_{\min})) + s_2(x_2(m_{\min}))$ and

   (b) $\dfrac{d}{dx_2}(s_2(x_2(m_{\max}))) < 0$ in $[d_2, x_{2\max}]$. In other words, $(s_2(x_2(m_{\max}))$ is a decreasing function of $x_2$ in $[d_2, x_{2\max}]$.

# 3 Analyze Problems Geometrically

It follows from our discussion in Section 2 above, the optimal solution for a three ladders-walls problem occurs when one outsider ladder is vertical and it follows from Example 2 above that the second ladder could be resting on the tops of middle and third walls. In this section, we consider such scenario where the middle ladder resting on tops of the middle wall and one of the outside walls, and ask where to place the vertical ladder. For example, consider the Figure 9 below, we have two scenarios to consider, one is the total length $A'AER$ and the other is $S'SEP$. We would like to ask if we should place the vertical ladder on $AA'$ or $SS'$.

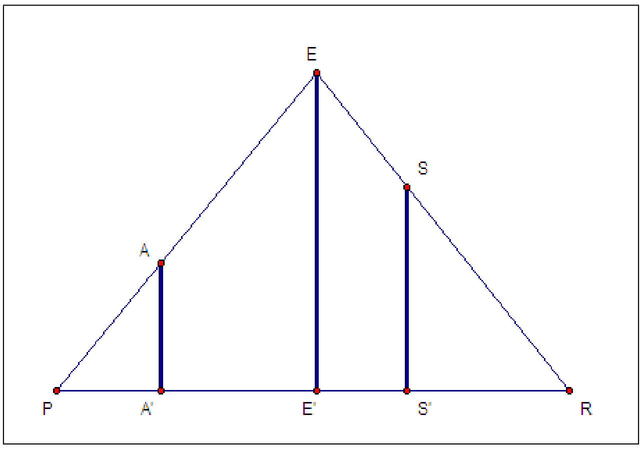



Figure 9

It is natural to consider the problems from geometric point of view once we know the solutions for three ladders-walls problems lie on the boundary. We divide various scenarios into several sub-problems. The example below shows that when $h_1$,$h_0$ and $d_1$ are given, the choice of selecting the vertical ladder on $AA'$ or $SS'$ does depend on the width $d_2$, and also on the height of $h_2$.

**Example 3** *Consider the three ladders-walls given by the Figure 10 below: We are given the heights of two walls $h_1 = GE, h_0 = CA$ with $h_1 < h_0$, and the width between $h_1$ and $h_0$, denoted by $d_1 = EA$, is fixed. Then there exists a point $F \in \overleftrightarrow{AB}$ (by dragging $F$ along $\overleftrightarrow{AB}$) such that one of the followings is true*

$$
\begin{aligned}
&BC + CH + HF < DC + CG + GE \text{ or} \\
&BC + CH + HF = DC + CG + GE \text{ or} \\
&BC + CH + HF > DC + CG + GE. \qquad (30)
\end{aligned}
$$

*In other words, the choice of selecting the vertical ladder on $GF$ or $HF$ depends on the width between the wall $CA$ and $HF$, and also the height of the wall $HF$.*

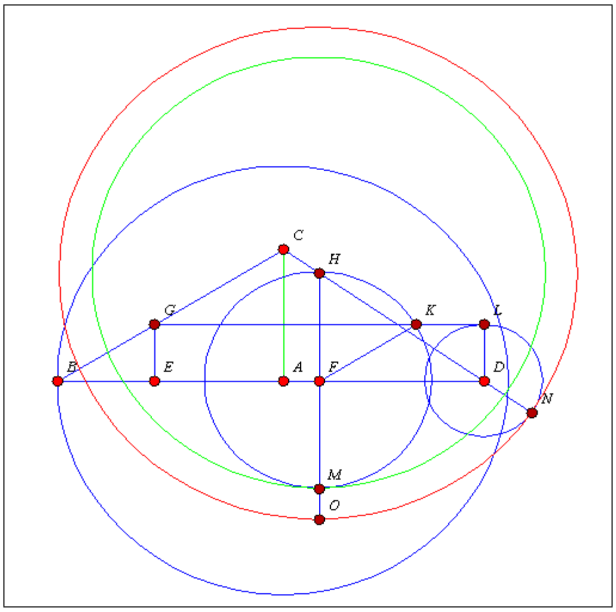



*Figure 10. Example 3*

We first explain our constructions as follows:
Step 1. We draw the parallelogram $BGKF$ so that $FK$ is parallel to $BG$ and $GK$ is parallel to $BF$.
Step 2. We construct a rectangle $GEDL$ so that $GE = LD$ and $GL = ED$.
Step 3. We construct the circle such that the center is at $F$ with radius $FK$ and note that $FK = FM$.
Step 4. We construct the circle such that the center is at $D$ with radius $DL$ and note that $DL = DN$.
Step 5. We construct the circle such that the center is at $H$ with radius $HN$; this is the circle shown in green color in Figure 10.
Step 6. We construct the circle such that the center is at $H$ with radius $HO$; this is the circle shown in red color in Figure 10.
We note

$$BC + CH + HF = BG + GC + CH + HF = BG + GCH + HF \tag{31}$$
$$DC + CG + GE = DH + HC + CG + GE = DH + HCG + GE \tag{32}$$

(31)-(32) yields,

$$(BG + HF) + GCH - ((DH + GE) + HCG) = (BG + HF) - (DH + GE) \tag{33}$$

since $GCH = HCG$. We note that

$$BG = FK \tag{34}$$

since $BGKF$ is a parallelogram We see that

$$GE = DL \tag{35}$$

since $EGLD$ is rectangle. We set

$$FK = FM \tag{36}$$

using the circle where $F$ is center and $FK$ is radius.
We set

$$DL = DN \tag{37}$$

using the circle where $D$ is center and $DL$ is radius. It follows from (34) and (36) that

$$BG = FM, \tag{38}$$

and it follows from (35) and (37) that

$$GE = DN \tag{39}$$

We substitute (38) and (39) into (33) to obtain

$$\begin{aligned}(BG + HF) - (DH + GE) &= (FM + HF) - (DH + DN)\\ &= HM - HN = HM - HO\end{aligned}$$

where we set $HN = HO$ by using the circle (in red color) described in Step 6 . In other words, the choice of a vertical or grounded ladder on wall $h_2$(or $HF$) of achieving shortest total length depends if $HM - HO$ is positive or negative:
**Case 1.** If $HM - HO < 0$ (or the green circle is inside the red circle), then $(BC + CH + HF) - (DC + CG + GE) < 0$, which means we should place the grounded ladder on the wall $GE$ or use $HF$ as the vertical ladder.
**Case 2.** If $HM - HO > 0$ (or the red circle is inside the green circle), then $(BC + CH + HF) - (DC + CG + GE) > 0$, which means we should place the grounded ladder on the wall $HF$ or use $GE$ as the vertical ladder.
**Case 3.** If $HM - HO = 0$, then it does not matter which way we place the grounded ladder, the total length will be the same.

**Remark:** In the Example 3 above, if we move $h_2$ toward $h_0$ (or move $HF$ toward $CA$), then the three ladders-walls problem is reduced to the two ladders-walls whose solution is consistent with the one described in [2].

The next Example shows that if we fix the heights of the walls $h_1$ and $h_0$, we further fix the widths of $d_1$ and $d_2$. Then we can determine the height of $h_2$ when choosing the vertical ladder on $h_1$ or $h_2$.

**Example 4** *Consider the three ladders-walls given in Figure 10: We are given the heights of two walls $h_1 = GE, h_0 = CA$ with $h_1 < h_0$, and the widths $d_1 = EA$ and $d_2 = AF$ are fixed. Then we can find the height for the third wall $h_2 = HF < h_0$ (by dragging $D$) so that one of the followings is true*

$$\begin{aligned}&BC + CH + HF < DC + CG + GE \text{ or}\\ &BC + CH + HF = DC + CG + GE \text{ or}\\ &BC + CH + HF > DC + CG + GE\end{aligned} \tag{40}$$

*By dragging the point $D$ on from Figure 10 (see [8] in the section of Supplemental Electronic Materials), we see three possibilities of $HM - HO > 0, HM - HO = 0$ and $HM - HO < 0$.*

The next example shows how we can achieve the equality in equation (40). In particular, given the heights of two walls $h_1$ and $h_0$ with $h_1 < h_0$; we fix widths $d_1$, we demonstrate a constructive way of determing the width $d_2$, between the walls $h_0$ and $h_2$, such that the total length of placing the vertical ladder on either outside wall is the same.

**Example 5** *Consider the following Figure 11, whre the heights of three walls, $DB$, $CA$ and $LM$ are given, and we fix the distance $(d_1)$ between the wall $DB$ and the wall $CA$, then the distance $(d_2)$ between $CA$ and $LM$ can be determined so that*

$$EC + CL + LM = NC + CD + DB. \tag{41}$$

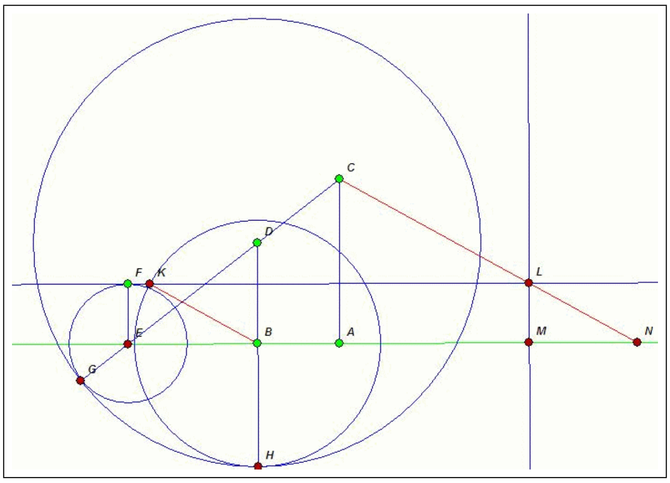



*Figure 11. Example 5*

*Step1: We draw the line $\overleftrightarrow{CD}$ and find the intersection between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, which we label it $E$.*
*Step 2: We draw a perpendicular segment $EF$ to $\overleftrightarrow{AB}$ at $E$; next we draw a line that is passing through $F$ and parallel to $\overleftrightarrow{AB}$. We call such line $L_F$.*
*Step 3. We use $E$ as the center and $FE$ as the radius to draw a circle, which gives us the point $G$, lying on the line $\overleftrightarrow{AB}$ and $GE = EF$.*
*Step 4. We use $D$ as the center and $DG$ as the radius to draw the circle, which gives us the point $H$, lying on the line $\overleftrightarrow{DB}$ and note that $DH = DG$. Also, we get the intersection between the circle and the line $L_F$ (from Step 2), which we denote the point by $K$.*
*Step 5. We connect $KB$ and construct the line that is passing through $C$ and is parallel to $KB$, and we call such line $L_C$. Consequently, we obtain the point $N$ which is the intersection between $L_C$ and $\overleftrightarrow{AB}$. Additionally, we obtain the point $L$ which is the intersection between $L_C$ and $L_F$.*
*Step 6. Finally, we draw the line passing through $L$ and is perpendicular to $\overleftrightarrow{AB}$ to get the point $M$.*
*We shall claim that the construction process described above yields*

$$EC + CL + LM = NC + CD + DB. \tag{42}$$

*We note*

$$(EC + CL + LM) - (NC + CD + DB) \tag{43}$$
$$= [(ED + DC) + CL + LM] - [(NL + LC) + CD + DB] \tag{44}$$
$$= (ED + LM) - (NL + DB). \tag{45}$$

*However,* $ED + LM = ED + GE = DG$ *and* $NL + DB = KB + DB = HB + BD = HD$. *We refer readers to [9] for detailed construction and demonstration.*

The next example demonstrates a special case of the Example 5, which shows the shortest total length will depend on the widths between two ladders.

**Example 6** *Consider the following three ladders-walls problem in Figure 12, where* $EP = ER, SS' > AA'$, *and* $A'E' > E'S'$, *we shall prove*

$$RE + EA + AA' > PE + ES + SS'. \tag{46}$$

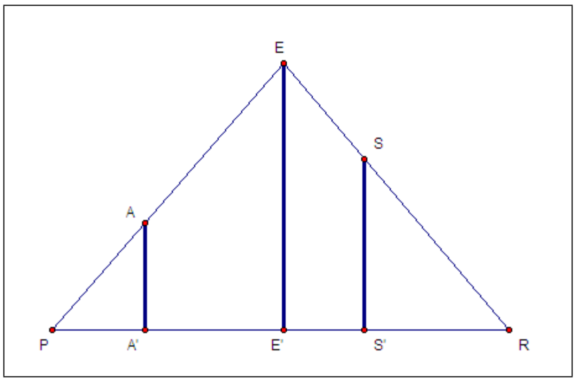



*Figure 12*

*We first add* $AU$ *so that* $AU$ *is parallel to* $PR$; *next we label the intersection between* $SS'$*and* $AU$ *by* $V$; *we draw a line passing through* $V$, *parallel to* $ER$ *and intersects* $PR$ *at* $W$ *as shown in Figure 13. We note that*

$$\begin{aligned} & RE + EA + AA' - PE - ES - SS' \\ & = EU + VS' - ES - SS' \\ & = SU - SV > 0, \end{aligned} \tag{47}$$

*when we consider the right triangle* $SUV$.

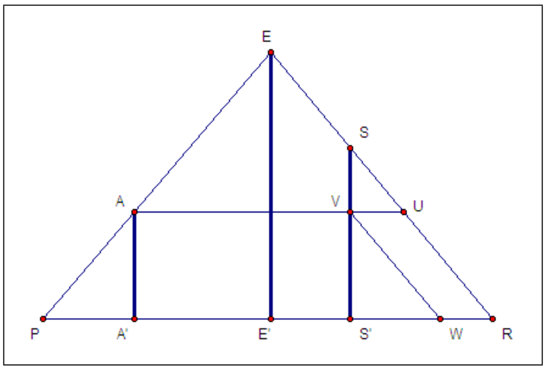



*Figure 13*

*We note that this provides a solution to the special case when* $PE = ER$. *The solution suggests that we should use grounded ladder on shorter wall* $AA'$ *in this case.*

**Remark:** We further note that if we decrease the length of the wall $SS'$ or let $S \to U$, the length of $SS'$will get close to the length of the wall $AA'$. In such case, we can see that $SU \to SV$; in other words, $RE + EA + AA' - PE - ES - SS' \to 0$. This consists with what we expected that when wall $h_1 = h_2$, the ground or vertical ladder can be from either end.

## 3.1 Extension to more ladders and walls

We introduce some terminologies:

- We only consider the $n-$ladders and $n-$walls system, denoted by $\{L_i, W_i\}_{i=1}^n$, when the number of walls is the same as the number of ladders. When no confusion occurs, we simply write $\{L_i\}_{i=1}^n$ or call it a $n$-ladder system. For example, a four-ladder system could look like the following Figure 14.

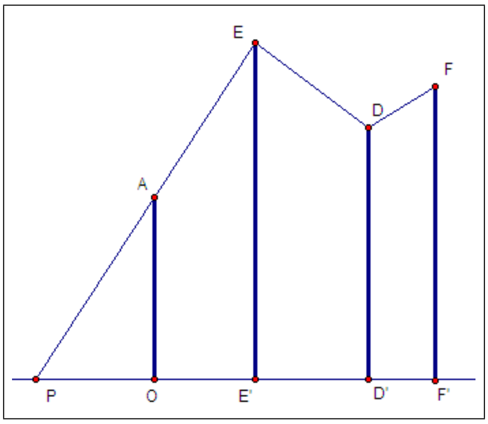



Figure 14

  In this case, the ladder $L_1 = EP$ is grounded (where $P$ is obtained by extending $EA$ to the ground level).

- We further note that in the four-ladder system in Figure 14, $L_1$ ($EP$) and $L_2$ ($ED$) have to touch the point $E$; in other words, $EP$ can not go over the point $E$, otherwise, it will not reach the optimal case. Similar conclusion can be drawn for $n$-ladder system.

- For $n = 7$, a seven-ladder system could look like the one in Figure 15 below:

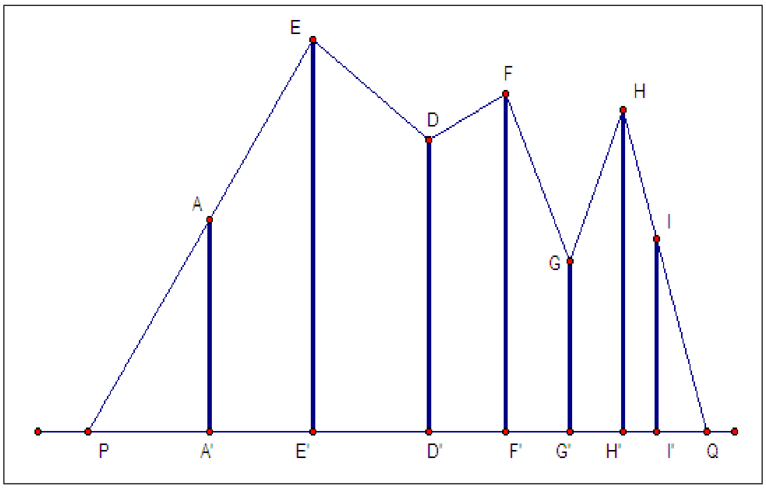



Figure 15

To find the shortest total length for the seven ladders-walls described in Figure 15.
Step 1. We only need to determine if we place the vertical on the left wall $AA'$ or the right wall $II'$ by comparing $AA' + IQ$ and $II' + AP$.
Step 2. Assume $AA' + IQ < II' + AP$. We choose $L_1 = AA'$ to be vertical and $L_2 = AE, L_3 = ED, L_4 = DF, L_5 = FG, L_6 = GH$, and $L_7 = HQ$, which is grounded. The solution to $AA' + IQ > II' + AP$ can be derived analogously.

# 4 Conclusion

Solving the three ladders-walls was first explored with the help of a dynamic geometry system ([ClassPad] in this case), and it was surprising to conjecture that the optimal solution exist near the boundary, either using a left vertical or right vertical ladder. After extensive computations with the help of a CAS, the optimal solutions turn out to be consistent with what we had conjectured when dynamic geometry software was used. We further explore the shortest total length for special cases, using dynamic geometry approach, by varying the given conditions on the heights of the walls or the widths between two consecutive walls in Section 3. Each example mentioned in Section 3 can be a separate problem that is accessible to even middle school students. For special cases of three ladders-walls problems discussed in Section 3, finding solutions geometrically is much more preferable than using a CAS for computation. Finally, we extend the results to finitely many ladders-walls systems by making some simple observations. In summary, the ladders-walls problems are interesting for students to explore as project based activities.

# 5 Acknowledgement

Authors would like to thank Hirotaka Ebisui of Japan for contributing the constructions demonstrated in Examples 3 and 4 in Section 3, and Douglas Meade for insightful discussion on this work.

# References

[1] D. H. Edwards, *Ladders, Moats, and Lagrange Multipliers*, The Mathematica Journal 4 (1994), pages 48-52.

[2] J. Gerlach, *Two Walls and Two Ladders - A Calculus Problem*, The Electronic Journal of Mathematics and Technology, Volume 3, Number 1, ISSN 1933-2823, 2009, pages 73-82.

[3] J. Stewart, *Single Variable Calculus (Early Transcendentals),* 5th ed., Thomson Brooks/Cole, 2003.

# 6 Software Packages and Supplemental Electronic Materials

[4] [Cinderella] A product of Cinderella I/S, http://www.cinderella.de/tiki-index.php.

[5] [ClassPad] A product of CASIO Computer Ltd., http://classpad.net or http://classpad.org/.

[6] [MAPLE] A product of MAPLEsoft, http://www.MAPLEsoft.com/.

[7] Ebisui, H., A *Cinderella* file for Example 3.

[8] Ebisui, H., A *Cinderella* file for Example 4.

[9] Ebisui, H., A *Cinderella* file for Example 5.

[10] Kawski, M., *MAPLE* computation for the three ladder-wall problem-Example 1.

[11] Kawski, M., *MAPLE* computation for the three ladder-wall problem-Example 2.